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Complete and almost complete minors in double-critical 8-chromatic graphs

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Abstract

A connected k-chromatic graph G is said to be *double-critical* if for all edges uv of G the graph $G - u - v$ is $(k-2)$-colourable. A long-standing conjecture of Erdős and Lovász states that the complete graphs are the only double-critical graphs. Kawarabayashi, Pedersen and Toft [*Electron. J. Combin.*, 17(1): Research Paper 87, 2010] proved that every double-critical k-chromatic graph with $k \leq 7$ contains a K_k minor. It remains unknown whether an arbitrary double-critical 8-chromatic graph contains a K_8 minor, but in this paper we prove that any double-critical 8-chromatic contains a K_8^- minor; here K_8^- denotes the complete 8-graph with one edge missing. In addition, we observe that any double-critical 8-chromatic graph with minimum degree different from 10 and 11 contains a K_8 minor.

1 Introduction and motivation

At the very center of the theory of graph colouring is Hadwiger's Conjecture which dates back to 1942. It states that every k-chromatic graph[1] contains a K_k minor.

Conjecture 1.1 (Hadwiger [10])**.** *If G is a k-chromatic graph, then G contains a K_k minor.*

Hadwiger [10] showed that the conjecture holds for $k \leq 4$, the case $k = 4$ being the first non-trivial instance of the conjecture. Later, several short and elegant proofs for the case $k = 4$ were found; see, for instance, [30]. The case $k = 5$ was studied independently by Wagner [31], who proved that the case $k = 5$ is equivalent to the Four Colour Problem. In the early 1960s, Dirac [7] and Wagner [32], independently, proved that every 5-chromatic graph G contains a

[1]All graphs considered in this paper are undirected, simple, and finite. The reader is referred to Section 3 for basic graph-theoretic terminology and notation.

K_5^- minor, that is, G contains, as a minor, a complete 5-graph with at most one edge missing. The case $k = 5$ of Hadwiger's Conjecture was finally settled in the affirmative with Appel and Haken's proof of the Four Colour Theorem [1, 2] (an improved proof was subsequently published in 1997 by Robertson et al. [25]). In 1964, Dirac [8] proved that every 6-chromatic graph contains a K_6^- minor (see [29, p. 257] for a short version of Dirac's proof), and, in 1993, Robertson, Seymour and Thomas [24] proved, using the Four Colour Theorem, that every 6-chromatic graph contains a K_6 minor. Thus, Hadwiger's Conjecture has been settled in the affirmative for each $k \leq 6$, but remains unsettled for all $k \geq 7$. In the early 1970s, Jakobsen [11, 12, 13] proved that for $k = 7, 8$, and 9 every k-chromatic graph contains, as a minor, K_7^{--}, K_7^-, and K_7, respectively, and these results seem to be the best obtained so far in support of Hadwiger's Conjecture for the cases $k = 7, 8$, and 9. (Here K_7^- denotes the complete 7-graph with one edge missing, while K_7^{--} denotes a complete 7-graph with two edges missing. There are two non-isomorphic complete 7-graphs with two edges missing.) The interested reader is referred to [14, 30] for a thorough survey of Hadwiger's Conjecture and related conjectures.

Another longstanding conjecture in the theory of graph colouring is the so-called Erdős-Lovász Tihany Conjecture which dates back to 1966. This conjecture states, in an interesting special case, that the complete graphs are the only double-critical graphs [9]. A connected k-chromatic graph G is *double-critical* if for all edges uv of G the graph $G - u - v$ is $(k-2)$-colourable.

Conjecture 1.2 (Erdős & Lovász [9]). *If G is a double-critical k-chromatic graph, then G is isomorphic to K_k.*

Conjecture 1.2, which we call the *Double-Critical Graph Conjecture*, is settled in the affirmative for all $k \leq 5$, but remains unsettled for all $k \geq 6$ [22, 27, 28]. As a relaxed version of the Double-Critical Graph Conjecture the following conjecture was posed in [17].

Conjecture 1.3 (Kawarabayashi, Pedersen & Toft [17]). *If G is a double-critical k-chromatic graph, then G contains a K_k minor.*

Conjecture 1.3 is, of course, also a relaxed version of Hadwiger's Conjecture, and so we call it the *Double-Critical Hadwiger Conjecture*; in [17], it was settled in the affirmative for $k \in \{6, 7\}$ (without use of the Four Colour Theorem) but it remains open for all $k \geq 8$. Very little seems to be known about complete minors in 8-chromatic graphs. The best result so far in the direction of proving the Hadwiger Conjecture for 8-chromatic graphs seems to be a theorem published in 1970 by Jakobsen [11]; the theorem states that every 8-chromatic graph contains a K_7^- minor. Corollary 7.3 in [17] states that every double-critical k-chromatic graph with $k \geq 7$ contains a K_7 minor. In this paper we prove that every double-critical 8-chromatic graph contains a K_8^- minor. The proof of this result is surprisingly complicated and uses a number of deep results by other authors.

2 Main results

These are our main results.

Theorem 2.1. *Every double-critical* 8*-chromatic graph contains a* K_8^- *minor.*

Corollary 2.2. *Every double-critical* k*-chromatic graph with* $k \geq 8$ *contains a* K_8^- *minor.*

In the case of minimum degree different from 10 and 11 we are able to find 'the edge missing in Theorem 2.1'.

Theorem 2.3. *Every double-critical* 8*-chromatic graph with minimum degree different from* 10 *and* 11 *contains a* K_8 *minor.*

Our proofs of the above-mentioned results do not rely on the Four Colour Theorem but they do rely on the following two deep results.

Theorem 2.4 ((i) Song [26]; (ii) Jørgensen [15])**.** *Suppose* G *is a graph on at least* 8 *vertices.*

(i) *If* G *has more than* $\lceil (11n(G) - 35)/2 \rceil$ *edges, then* G *contains a* K_8^- *minor, and*

(ii) *if* G *has more than* $6n(G) - 20$ *edges, then* G *contains a* K_8 *minor.*

Proof of Theorem 2.3. Suppose G is a double-critical 8-chromatic graph with minimum degree $\delta(G)$. Then, according to Proposition 3.1 (ii), $\delta(G) \geq 9$. If $\delta(G) \geq 12$, then $|E(G)| \geq 6n(G)$ and so, by Theorem 2.4 (ii), $G \geq K_8$. If $\delta(G) = 9$, then the desired result follows from Corollary 4.2. □

Proof of Theorem 2.1. Let G denote a double-critical 8-chromatic graph. By Theorem 2.3, we may assume $\delta(G) \geq 10$. If $\delta(G) \geq 11$, then $|E(G)| \geq 11n(G)/2$ and so, by Theorem 2.4 (i), $G \geq K_8^-$. Suppose $\delta(G) = 10$, let x denote a vertex of degree 10 in G, and define $G_x := G[N(x)]$. Then, according to Observation 5.1, $\Delta(\overline{G_x}) \leq 3$. If $\Delta(\overline{G_x}) \leq 2$, then, by Proposition 5.2, $G \geq K_8$. If $\Delta(\overline{G_x}) = 3$ and G_x contains at least one vertex of degree 9, then, by Proposition 6.2, $G \geq K_8^-$. If $\Delta(\overline{G_x}) = 3$ and G_x contains no vertex of degree 9, then, by Proposition 6.4, $G \geq K_8^-$. This completes the proof. □

Proof of Corollary 2.2. Let G denote a double-critical k-chromatic graph with $k \geq 8$. If $k = 8$ or $\delta(G) \geq 11$, then the desired result follows from Theorem 2.1 or Theorem 2.4 (i), respectively. Hence, by Proposition 3.1 (ii), we may assume $k = 9$ and $\delta(G) = 10$; in this case we prove $G \geq K_8^-$ by an argument somewhat similar to the first part of the proof of Proposition 4.1. The details are omitted. □

3 Preliminaries and notation

We shall use standard graph-theoretic terminology and notation as defined in [4, 6] with a few additions. Given any graph G, $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set, while $\overline{G}$ denotes the complement of G. The *order* of a graph G, that is, the number of vertices in G, is denoted $n(G)$, and any graph on n vertices is called an *n-graph*. A vertex of degree k in a graph G is said to be a *k-vertex* (of G). Given two graphs H and G, the *complete join* of G and H, denoted $G+H$, is the graph obtained from two vertex-disjoint copies of H and G by joining each vertex of the copy of G to each vertex of

the copy of H. For every positive integer k and graph G, kG denotes the graph $\sum_{i=1}^{k} G$. Given any edge-transitive graph G, any graph, which can be obtained from G by removing one edge, is denoted G^-. The *girth* of a graph G is the length of a shortest cycle in G; if G is acyclic, then the girth of G is said to be infinite. Given any subset X of the vertex set $V(G)$ of a graph G, we let $G[X]$ denote the subgraph of G induced by the vertices of X. The set of vertices of G adjacent to v is called the *neighbourhood of* v (in G), and it is denoted $N_G(v)$ or $N(v)$. The set $N(v) \cup \{v\}$ is called the *closed neighbourhood of* v (in G), and it is denoted $N_G[v]$ or $N[v]$. The induced graph $G[N(v)]$ is referred to as the *neighbourhood graph of* v (w.r.t. G), and it is denoted G_v. Given two graphs G and H, we say that H is a *minor* of G (and that G has an H *minor*) if there is a collection $\{V_h \mid h \in V(H)\}$ of non-empty, disjoint subsets of $V(G)$ such that the induced graph $G[V_h]$ is connected for each $h \in V(H)$, and for any two adjacent vertices h_1 and h_2 in H there is at least one edge in G joining some vertex of V_{h_1} to some vertex of V_{h_2}. The sets V_h are called the *branch sets* of the minor H of G. We may write $H \leq G$ or $G \geq H$, if G contains an H minor. In [17], a number of basic results on double-critical graphs were determined. We will make repeated use of these results and so, for ease of reference, they are restated here.

In the remaining part of this section, we let G denote a non-complete double-critical k-chromatic graph with $k \geq 6$. Given any edge $xy \in E(G)$, define

$$\begin{aligned} A(x,y) &:= N(x) \setminus N[y] \\ B(x,y) &:= N(x) \cap N(y) \\ C(x,y) &:= N(y) \setminus N[x] \end{aligned}$$

Proposition 3.1 ([17]).

(i) *The graph G does not contain a complete $(k-1)$-graph as a subgraph;*

(ii) *the graph G has minimum degree at least $k+1$, and*

(iii) *for all edges $xy \in E(G)$ and all $(k-2)$-colourings of $G - x - y$, the set $B(x,y)$ of common neighbours of x and y in G contains vertices from every colour class, in particular, $|B(x,y)| \geq k-2$.*

Proposition 3.2. *If $G[A(x,y)]$ is a complete graph for some edge $xy \in E(G)$, then there is a matching of the vertices of $A(x,y)$ to the vertices of $B(x,y)$ in $\overline{G_x}$.*

Proof. Suppose $G[A(x,y)]$ is a complete graph for some edge $xy \in E(G)$, and let $G - x - y$ be coloured properly in the colours $1, 2, \ldots, k-3$, and $k-2$. The colours applied to $A(x,y)$ are all distinct, and so we may assume $A(x,y) = \{a_1, \ldots, a_p\}$ where vertex a_i is coloured i for each $a_i \in A(x,y)$. According to Proposition 3.1 (iii), each of the colours $1, 2, \ldots, k-3$, and $k-2$ appear at least once on a vertex of $B(x,y)$, say $B(x,y) = \{b_1, \ldots, b_q\}$ with vertex b_i being coloured i for each $i \leq k-2$. Also, $q \geq k-2$. Since $G[A(x,y) \cup \{x\}]$ is a complete graph, it follows from Proposition 3.1 (i) that $p = |A(x,y)| \leq k-3$. Hence $p < q$, and a_i and b_i have the same colour for each $i \in [p]$, in particular, $\{a_1b_1, a_2b_2, \ldots, a_pb_p\}$ is a matching of the vertices of $A(x,y)$ to vertices of $B(x,y)$ in $\overline{G_x}$. $\square$

Proposition 3.3 ([17])**.** *If $A(x,y)$ is non-empty for some edge $xy \in E(G)$, then $\delta(G[A(x,y)]) \geq 1$, that is, the induced subgraph $G[A(x,y)]$ contains no isolated vertices. By symmetry, $\delta(G[C(x,y)]) \geq 1$, if $C(x,y)$ is non-empty.*

Thus, by Proposition 3.3, if y is a vertex which has degree 2 in $\overline{G_x}$ then the two neighbours of y in $\overline{G_x}$ must be non-adjacent in $\overline{G_x}$.

Proposition 3.4 ([17])**.**

(i) *For any vertex x of G not joined to all other vertices of G, $\chi(G_x) \leq k-3$;*

(ii) *if x is a vertex of degree $k+1$ in G, then the complement $\overline{G_x}$ consists of isolated vertices (possibly none) and cycles (at least one), where the length of each cycle is at least five, and*

(iii) *G is 6-connected.*

Proposition 3.5 ([17])**.** *There is no non-complete double-critical 8-chromatic graph of order less than 15.*

4 Minimum degree 9 and K_8 minors

Proposition 4.1. *If G is a double-critical 8-chromatic graph with a vertex x of degree 9, then $G_x \simeq \overline{C_8} + K_1$ or $G_x \simeq \overline{C_9}$.*

Proof. Suppose G is a double-critical 8-chromatic graph with a vertex x of degree 9. Now, according to Proposition 3.4 (ii), $\overline{G_x}$ consists of isolated vertices and cycles (at least one cycle) of length at least 5. Since G_x consists of only nine vertices, it follows that $\overline{G_x}$ consists of exactly one cycle, which we denote C_j, and some isolated vertices. If $j \in \{5,6\}$, then $G[N[x]]$ is easily seen to contain K_7 as a subgraph, contrary to Proposition 3.1 (i). Suppose $j = 7$. Moreover, suppose that the vertex x is not adjacent to all other vertices of G. Then, according to Proposition 3.4 (i), $\chi(G_x) \leq 5$. However, the graph G_x, which is isomorphic to $\overline{C_7} + K_2$, is easily seen not be 5-colourable. Thus, the vertex x is adjacent to all other vertices of G, and so G is isomorphic to $\overline{C_7} + K_3$. However, the graph $\overline{C_7} + K_3$ is easily seen to be 7-colourable, a contradiction. Thus, we must have $j \geq 8$, and so the desired result follows immediately. □

The proof of Proposition 4.1 implies that any double-critical 8-chromatic graph with a vertex of degree 9 contains K_6^- as a subgraph.

Corollary 4.2. *Every double-critical 8-chromatic graph with minimum degree 9 contains a K_8 minor.*

Proof. Suppose G is a double-critical 8-chromatic graph with minimum degree 9, and let x denote a vertex of G of degree 9. Suppose that G does not contain a K_8 minor. Then, according to Proposition 3.5, there are at least 15 vertices in G, in particular, there is a vertex, which we shall call z, in $G - N[x]$. According to Proposition 4.1, there are two cases to consider: either $G_x \simeq \overline{C_8} + K_1$ or $G_x \simeq \overline{C_9}$.

Suppose $G_x \simeq \overline{C_8} + K_1$, where $C_8 : v_0, v_1, v_2, \ldots, v_7$ and $V(K_1) = \{u\}$. By Proposition 3.4 (iii), G is 6-connected, and so $G - u$ must be 5-connected. Now, according to Menger's Theorem (see, for instance, [4, Theorem 9.1]), there is a

collection $\mathcal{C}$ of five internally vertex-disjoint (x,z)-paths in $G-u$. Obviously, each path $P \in \mathcal{C}$ contains a vertex from $V(C_8)$, and we may assume that each of the paths $P \in \mathcal{C}$ contains exactly one vertex from $V(C_8)$. The fact that there are eight vertices in $V(C_8)$ and five vertex-disjoint (x,z)-paths in $\mathcal{C}$ going through $V(C_8)$ implies the existence of a pair of vertices v_i and v_{i+1} (modulo 8) such that there is a (v_i,z)-path Q_i and a (v_{i+1},z)-path Q_{i+1} in $G-u$ such that Q_i and Q_{i+1} are internally vertex-disjoint. We may assume $i=0$. Now, the (v_0,v_1)-path $Q_0 \cup Q_1$ in G is contracted to an edge between v_0 and v_1. The resulting graph contains the graph $H \simeq \overline{C_8^-} + K_2$ as a subgraph, and H can be contracted to K_8 by contracting the edges v_2v_5 and v_4v_7. Thus, $G \geq K_8$. A similar argument shows that, if $G_x \simeq \overline{C_9}$, then $G \geq K_8$. □

5 Minimum degree 10 and K_8 minors

Observation 5.1. *If G is a double-critical 8-chromatic graph with minimum degree 10 and $\deg(x,G)=10$, then $\Delta(\overline{G_x}) \leq 3$.*

Proof. Suppose $\Delta(\overline{G_x}) \geq 4$, and let y denote a vertex which has degree ≥ 4 in $\overline{G_x}$. Then $|A(x,y)| \geq 4$ and, according to Proposition 3.1 (iii), $|B(x,y)| \geq 6$. Thus, $\deg(x,G) \geq |A(x,y)| + |B(x,y)| + 1 \geq 11$, which contradicts the assumption $\deg(x,G)=10$. □

Proposition 5.2. *Suppose G is a double-critical 8-chromatic graph with minimum degree 10, and suppose G contains a vertex x of degree 10 such that $\Delta(\overline{G_x}) \leq 2$. Then G contains a K_8 minor.*

Proof. If $\Delta(\overline{G_x}) = 0$, then $G_x \simeq K_{10}$, a contradiction. According to Proposition 3.3, no vertex of $\overline{G_x}$ has degree exactly 1. Hence, $\Delta(\overline{G_x}) = 2$, and so the graph $\overline{G_x}$ consists of cycles (at least one) and possibly some isolated vertices. If $\overline{G_x}$ has at least five isolated vertices, then it is easy to see that G_x contains K_7 as a subgraph. If $\overline{G_x}$ has exactly four isolated vertices then either $G_x \simeq K_4 + 2\overline{K_3}$ or $G_x \simeq K_4 + \overline{C_6}$. In the former case we obtain $G_x \geq K_7$ and in the latter case $G_x \supset K_7$. If $\overline{G_x}$ has exactly three isolated vertices, then either $G_x \simeq K_3 + \overline{C_3} + \overline{C_4}$ or $G_x \simeq K_3 + \overline{C_7}$. If $\overline{G_x}$ has exactly two isolated vertices, then G_x is isomorphic to either $K_2 + \overline{K_3} + \overline{C_5}$, $K_2 + 2\overline{C_4}$, or $K_2 + \overline{C_8}$. If $\overline{G_x}$ has exactly one isolated vertices, then G_x is isomorphic to either $K_1 + 3\overline{K_3}$, $K_1 + \overline{K_3} + \overline{C_6}$, $K_1 + \overline{C_4} + \overline{C_5}$, or $K_1 + \overline{C_9}$. If $\overline{G_x}$ has no isolated vertices, then G_x is isomorphic to either $2\overline{K_3} + \overline{C_4}$, $\overline{K_3} + \overline{C_7}$, $\overline{C_4} + \overline{C_6}$, $2\overline{C_5}$, or $\overline{C_{10}}$. In each case it is easy to exhibit a K_7 minor in G_x, and so $G \geq K_8$. □

It may be true that if G is a double-critical 8-chromatic graph with minimum degree 10 and a vertex x of degree 10 such that $G[N(x)]$ is 6-regular then G contains a K_8 minor. I was only able to prove the desired result when $\overline{G[N(x)]}$ is not isomorphic to any of the eight graphs $G_7, G_8, G_9, G_{12}, G_{13}, G_{16}, G_{17}$, and G_{19} (see Appendix A). The graph denoted G_{17} is the Petersen graph. Given the symmetry of the Petersen graph, it is particularly annoying not being able to settle the case $\overline{G[N(x)]} \simeq G_{17}$.

Problem 5.3. *Prove that if G is a double-critical 8-chromatic graph with minimum degree 10 and a vertex x of degree 10 such that $\overline{G_x}$ is the Petersen graph, then G contains a K_8 minor.*

6 Minimum degree 10 and K_8^- minors

In this section, we shall apply the following result of Mader.

Theorem 6.1 (Mader [19]). *Every graph with minimum degree at least* 5 *contains* K_6^- *or the icosahedron graph as a minor. In particular, every graph with minimum degree at least* 5 *and at most* 11 *vertices contains a* K_6^- *minor.*

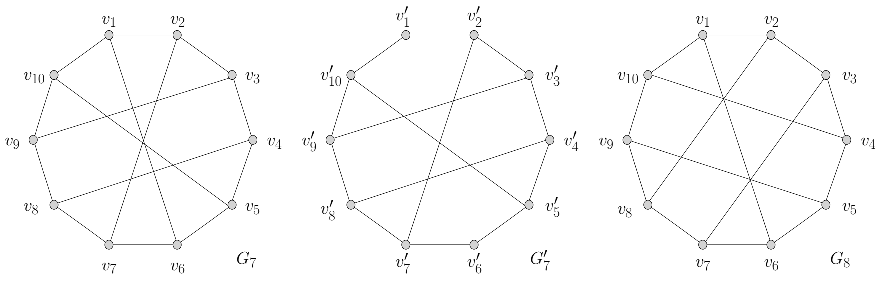
A proof of Theorem 6.1 may also be found in [3, p. 373].

Proposition 6.2. *Suppose* G *is a double-critical* 8*-chromatic graph with minimum degree* 10*. If* G *contains a vertex* x *of degree* 10 *such that* G_x *contains at least one vertex of degree* 9 *in* G_x*, then* G *contains a* K_8^- *minor.*

Proof. Suppose G is a double-critical 8-chromatic graph with minimum degree 10 such that a vertex, say v, has degree 9 in G_x. According to Observation 5.1, $\Delta(\overline{G_x}) \leq 3$ and so $\delta(G_x) = n(G_x) - 1 - \Delta(\overline{G_x}) \geq 6$. Thus, the graph $G_x - v$ has minimum degree at least 5 and exactly 9 vertices, and so it follows from Theorem 6.1 that $G_x - v$ contains a K_6^- minor. Such a K_6^- minor of $G_x - v$ along with the additional branch sets $\{x\}$ and $\{v\}$ constitute a K_8^- minor of G. □

Lemma 6.3. *Suppose* G *is a graph with a vertex* x *of degree* 10 *such that* $\overline{G_x}$ *is connected and cubic. Moreover, suppose that there is a vertex* $z \in V(G) \setminus N_G[x]$ *such that* G *contains at least six internally vertex-disjoint* (x, z)*-paths. Then* G *contains a* K_8^- *minor.*

Proof. Suppose G is a 6-connected graph with a vertex x of degree 10, where $\overline{G_x}$ is a connected cubic graph. There are exactly 21 non-isomorphic cubic graphs of order 10, see, for instance, [23]. These 21 non-isomorphic cubic graphs of order 10 are depicted in Appendix A; let these graphs be denoted as in Appendix A. If $\overline{G_x} \simeq G_i$, where $i \in [19] \setminus \{7, 8, 9, 12, 17\}$, then the labelling of the vertices of the graph G_i indicates how $\overline{G_i}$ may be contracted to K_7^- or K_7. The vertices labelled $j \in [7]$ constitute the jth branch set of a K_7^- minor or K_7 minor. If the branch sets only constitute a K_7^- minor, then it is because there is no edge between the branch sets of vertices labelled 1 and 7, respectively. In order to



(a) The graph G_7. (b) The graph G'_7. (c) The graph G_8.

Figure 1: The graphs G_7, G'_7, and G_8, which occur in the cases (i) and (ii) in the proof of Lemma 6.3.

handle the cases $\overline{G_x} \simeq G_i$, where $i \in \{7, 8, 9, 12, 17\}$, we use the assumption that $V(G) \setminus N_G[x]$ contains a vertex z such that G has a collection $\mathcal{R}$ of at least six internally vertex-disjoint (x, z)-paths.

(i) Suppose $\overline{G_x} \simeq G_7$ with the vertices of $\overline{G_x}$ labelled as shown in Figure 1 (a). Let $\mathcal{S}$ denote the collection of the five 2-sets $\{v_1, v_6\}$, $\{v_2, v_7\}$, $\{v_3, v_9\}$, $\{v_4, v_8\}$ and $\{v_5, v_{10}\}$. Since the 2-sets in $\mathcal{S}$ are pairwise disjoint and cover $N_G(x)$, it follows from the pigeonhole principle that at least two of the internally vertex-disjoint (x, z)-paths, say Q_1 and Q_2, of $\mathcal{R}$ go through the same 2-set $S \in \mathcal{S}$. If $S = \{v_i, v_j\} \in \mathcal{S} \setminus \{\{v_1, v_6\}\}$, then, by contracting the (v_i, v_j)-path $(Q_1 \cup Q_2) - x$ into the edge v_iv_j, we obtain a graph which, as is readily verifiable, has a K_7^- minor in the neighbourhood of x and so $G \geq K_8^-$. Hence, we may assume that $\mathcal{R}$ contains no such two paths going through the same 2-set of $\mathcal{S} \setminus \{\{v_1, v_6\}\}$. Hence $S = \{v_1, v_6\}$ with say Q_1 and Q_2 going through v_1 and v_6, respectively. Since $|\mathcal{R}| \geq 6$, there is precisely one path going through each of the sets $S' \in \mathcal{S} \setminus \{\{v_1, v_6\}\}$. By symmetry of $\overline{G_x}$, we may assume that there is an (x, z)-path $Q_3 \in \mathcal{R}$ going through the vertex v_2 of $N_G(x)$. Now, by contracting the (v_2, z)-path $Q_3 - x$ and the (v_6, z)-path $Q_2 - x$ into two edges, and then contracting the (v_1, z)-path $Q_1 - x$ into one vertex, we obtain a graph G' in which the neighbourhood graph $G'[N_G(x)]$ of x contains the complement of the G_7', depicted in Figure 1 (b), as a subgraph. The branch sets $\{v_1'\}$, $\{v_2'\}$, $\{v_3', v_5'\}$, $\{v_4', v_9'\}$, $\{v_6'\}$, $\{v_7', v_{10}'\}$, $\{v_8'\}$ constitute a K_7^- minor in $\overline{G_7'}$ (there may be no edge between the branch sets $\{v_8'\}$ and $\{v_4', v_9'\}$), and so $G \geq K_8^-$.

(ii) Suppose $\overline{G_x} \simeq G_8$ with the vertices of $\overline{G_x}$ labelled as shown in Figure 1 (c). In this case we contract a path $(P \cup Q) - x$, where $P, Q \in \mathcal{R}$, into an edge $e \in \{v_1v_6, v_2v_8, v_3v_7, v_4v_{10}, v_5v_9\}$, which is missing in G_x. By the symmetry of G_x, we need only consider the cases $e = v_1v_6$ and $e = v_2v_8$. If $e = v_1v_6$, then the branch sets $\{v_1, v_5\}$, $\{v_2\}$, $\{v_3, v_9\}$, $\{v_4, v_7\}$, $\{v_6\}$, $\{v_8\}$, and $\{v_{10}\}$ constitute a K_7^- minor in the neighbourhood of x. If $e = v_2v_8$, then the branch sets $\{v_1, v_9\}$, $\{v_2\}$, $\{v_3, v_6\}$, $\{v_4, v_7\}$, $\{v_5\}$, $\{v_8\}$, and $\{v_{10}\}$ constitute a K_7^- minor in the neighbourhood of x. In both cases we obtain $G \geq K_8^-$.

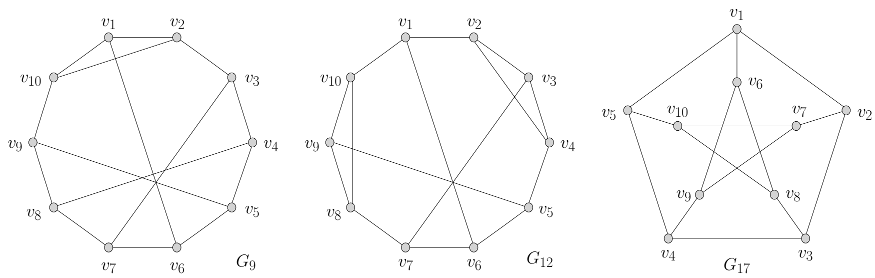
(iii) Suppose $\overline{G_x} \simeq G_9$ with the vertices of $\overline{G_x}$ labelled as shown in Figure 2 (a). Just as in case (ii), we contract a path $(P \cup Q) - x$, where $P, Q \in \mathcal{R}$, into an edge $e \in \{v_1v_6, v_2v_{10}, v_3v_7, v_4v_8, v_5v_9\}$. By the symmetry of G_x, we need only consider $e \in \{v_1v_6, v_2v_{10}, v_3v_7, v_4v_8\}$. If $e = v_1v_6$, then the branch sets $\{v_1\}$, $\{v_2, v_5\}$, $\{v_3\}$, $\{v_4, v_9\}$, $\{v_6\}$, $\{v_7, v_{10}\}$, and $\{v_8\}$ constitute a K_7^- minor in the neighbourhood of x. If $e = v_2v_{10}$, then the branch sets $\{v_1, v_8\}$, $\{v_2\}$, $\{v_3, v_5\}$, $\{v_4\}$, $\{v_6, v_9\}$ $\{v_7\}$, and $\{v_{10}\}$ constitute a K_7^- minor in the neighbourhood of x. If $e = v_3v_7$, then the branch sets $\{v_1, v_8\}$, $\{v_2, v_6\}$, $\{v_3\}$, $\{v_4, v_{10}\}$, $\{v_5\}$, $\{v_7\}$, and $\{v_9\}$ constitute a K_7^- minor in the neighbourhood of x. If $e = v_4v_8$, then the branch sets $\{v_1\}$, $\{v_2, v_5\}$, $\{v_3, v_9\}$, $\{v_4\}$, $\{v_6\}$, $\{v_7, v_{10}\}$, and $\{v_8\}$ constitute a K_7^- minor in the neighbourhood of x. In each case we obtain $G \geq K_8^-$.

(iv) Suppose $\overline{G_x} \simeq G_{12}$ with the vertices of $\overline{G_x}$ labelled as in Figure 2 (b). Again, we contract a path $(P \cup Q) - x$, where $P, Q \in \mathcal{R}$, into an edge

$e \in \{v_1v_6, v_2v_4, v_3v_7, v_5v_9, v_8v_{10}\}$. By the symmetry of G_x, we need only consider the cases $e \in \{v_1v_6, v_2v_4, v_3v_7\}$. If $e = v_1v_6$, then the branch sets $\{v_1\}$, $\{v_2, v_7\}$, $\{v_3\}$, $\{v_4, v_{10}\}$, $\{v_5, v_9\}$, $\{v_6\}$, and $\{v_8\}$ constitute a K_7^- minor in the neighbourhood of x. If $e = v_2v_4$, then the branch sets $\{v_1, v_5\}$, $\{v_2\}$, $\{v_3, v_8\}$, $\{v_4\}$, $\{v_6\}$, $\{v_7, v_{10}\}$, and $\{v_9\}$ constitute a K_7^- minor in the neighbourhood of x. If $e = v_3v_7$, then the branch sets $\{v_1, v_9\}$, $\{v_2, v_6\}$, $\{v_3\}$, $\{v_4, v_8\}$, $\{v_5\}$, $\{v_7\}$, and $\{v_{10}\}$ constitute a K_7^- minor in the neighbourhood of x. In each case we obtain $G \geq K_8^-$.

(v) Suppose $\overline{G_x} \simeq G_{17}$ with the vertices of $\overline{G_x}$ labelled as shown in Figure 2 (c). The graph G_{17} is the Petersen graph, and the complement of the Petersen graph does not contain a K_7 minor. However, we may repeat the trick used in the previous cases to obtain a K_7^- minor. We contract a path $(P \cup Q) - x$, where $P, Q \in \mathcal{R}$, into an edge $e \in \{v_iv_{i+5} \mid i \in [5]\}$. By the symmetry of G_x, we may assume $e = v_1v_6$. Now, the branch sets $\{v_1\}$, $\{v_2, v_8\}$, $\{v_3\}$, $\{v_4, v_{10}\}$, $\{v_5, v_9\}$, $\{v_6\}$, and $\{v_7\}$ constitute a K_7^- minor in the neighbourhood of x. Thus, G contains a K_8^- minor.

This completes the proof. □



(a) The graph G_9. (b) The graph G_{12}. (c) The graph G_{17}.

Figure 2: The graphs G_9, G_{12}, and G_{17}, which occur in the cases (iii), (iv), and (v) in the proof of Lemma 6.3.

Notice that in each of the cases (i-v) in the proof of Lemma 6.3 we used the regularity of G_x and the six internally vertex-disjoint (x, z)-paths of G, but we did not assume G to be double-critical. It may be possible to relax the assumptions of Lemma 6.3 and still maintain the conclusion. It may even be that Lemma 6.3 follows from an earlier result similar in spirit to that of Theorem 6.1.

Proposition 6.4. *Suppose G is a double-critical 8-chromatic graph with minimum degree 10. If G contains a vertex x of degree 10 such that G_x contains no vertex of degree 9 in G_x, then G contains a K_8^- minor.*

Proof. Suppose G is a double-critical 8-chromatic graph with minimum degree 10, and suppose G contains a vertex x of degree 10 such that G_x contains no vertex of degree 9 in G_x. Then it follows from Proposition 3.3 and Observation 5.1 that each vertex of $\overline{G_x}$ has degree 2 or 3.

We first consider the case where $\overline{G_x}$ is disconnected. Since $\delta(\overline{G_x}) \geq 2$, it follows that any component of $\overline{G_x}$ contains at least three vertices. If $\overline{G_x}$ contains a component on three vertices, then this component is a K_3; this contradicts Proposition 3.3. Hence, each component of $\overline{G_x}$ contains at least four vertices, and so, since $n(G_x) = 10$, it follows that $\overline{G_x}$ contains precisely two components, say D_1 and D_2 with $n(D_1) \leq n(D_2)$. Suppose $n(D_1) = 4$. The fact that $\delta(\overline{G_x}) \geq 2$ implies that D_1 must contain a 4-cycle, and so it is easy to see that D_1 must be C_4, K_4^- or K_4. This, however, contradicts Proposition 3.3, and so we must have $n(D_1) = n(D_2) = 5$. Of course, if G' is a subgraph of G, and G' contains an H minor, then G contains an H minor. Thus, it suffices to consider the case where both D_1 and D_2 contain exactly one vertex of degree 2, in which case both D_1 and D_2 is isomorphic to K_4 with exactly one edge subdivided. In this case it is very easy to find a K_7 minor in G_x.

Suppose that $\overline{G_x}$ is connected, and let D denote $\overline{G_x}$. By Proposition 3.5, we may assume there is a vertex $z \in V(G) \setminus N_G[x]$, and, by Proposition 3.4 (iii), there are six internally vertex-disjoint (x,z)-paths in G. If D is cubic, then, according to Lemma 6.3, $G \geq K_8^-$. Suppose that D is not cubic. We add edges (possibly none!) between non-adjacent 2-vertices to D to obtain D', which contains no two non-adjacent 2-vertices. If D' is cubic, then $G' := G \setminus (E(D') \setminus E(D))$ satisfies the assumption of Lemma 6.3. (The graph D' is connected, cubic 10-graph and the graph G' has six internally vertex-disjoint (x,z)-paths, since G has six internally vertex-disjoint (x,z)-paths, and these may be chosen so that they do not contain any edge of $E(G[N_G(x)])$.) Thus, $G' \geq K_8^-$, which implies that the supergraph G of G' has a K_8^- minor.

Now, suppose D' is not cubic. The graph D' contains no two non-adjacent 2-vertices. Moreover, D' is a connected 10-graph in which each vertex has degree 2 or 3. Thus, since the number of odd degree vertices of any graph is even it follows that D' contains exactly two 2-vertices and these must be neighbours. There are exactly 23 connected 10-graphs each with two 2-vertices and eight 3-vertices, where the two 2-vertices are adjacent[2]. These graphs, denoted J_i ($i \in [23]$), are depicted in Appendix B. For each $i \in [23]$, the labelling of the vertices of the graph J_i indicates how $\overline{J_i}$ may be contracted to K_7^- or, even, K_7; the vertices labelled $j \in [7]$ constitute the jth branch set of a K_7^-- or K_7 minor. If the branch sets only constitute a K_7^- minor, then it is because there is no edge between the branch sets labelled 1 and 7. This completes the proof. □

7 More open problems

The Double-Critical Graph Conjecture is still open for 6-chromatic graphs. To settle this instance of the conjecture in the affirmative, it would, by Proposition 3.1 (i), suffice to prove that any double-critical 6-chromatic graph contains K_5 as a subgraph; however, we cannot even prove that such a graph contains K_4 as a subgraph.

[2]According to the computer program **geng** developed by Brendan McKay [21], there are 113 connected graphs of order 10 each with two 2-vertices and eight 3-vertices – among these graphs exactly 23 have the property that the two 2-vertices are adjacent. This latter fact has been determined, independently, by inspection done by the author and by a computer program developed by Marco Chiarandini.

Problem 7.1 (Matthias Kriesell[3]). *Prove that every double-critical* 6*-chromatic graph contains* K_4 *as a subgraph.*

In [17], it was proved that every double-critical 6-chromatic graph contains a K_6 minor; a stronger result would be that every double-critical 6-chromatic graph contains a subdivision of K_6.

Problem 7.2. *Prove that every double-critical* 6*-chromatic graph* G *contains a subdivision of* K_6.

According to Observation 7.3, Problem 7.2 has a positive solution if G has minimum degree at most 7.

Mader [20] proved a longstanding conjecture, known as Dirac's Conjecture, which states that any graph G with at least three vertices and at least $3n(G)-5$ edges contains a subdivision of K_5. Thus, in particular, any double-critical 6-chromatic graph G contains a subdivision of K_5.

Observation 7.3. *Any double-critical* 6*-chromatic graph with minimum degree at most* 7 *contains a subdivision of* K_6.

Proposition 7.4 ([17]). *If* G *is a non-complete double-critical* 6*-chromatic graph, then* G *contains at least* 12 *vertices.*

Proof of Observation 7.3. Let G denote any double-critical 6-chromatic graph with minimum degree at most 7. If $\delta(G) \leq 6$, then, by Proposition 3.1 (i), $G \simeq K_6$. Hence $\delta(G) = 7$. Let x denote a vertex of degree 7 in G. The graph G is non-complete, and so, by Proposition 7.4, $n(G) \geq 12$, in particular, $G - N[x]$ is non-empty. Let z denote a vertex of $G - N[x]$. According to Corollary 6.1 in [17], $\overline{G_x}$ is a 7-cycle C_7 with, say, $C_7 : v_1, v_2, v_3, \ldots, v_7$. By Proposition 3.4 (iii), G is 6-connected, and so there is a collection $\mathcal{C} = \{Q_1, Q_2, \ldots, Q_6\}$ of six internally vertex (x, z)-paths in G. Choose the paths such that the sum of the lengths of the paths is minimum. Then each of the paths $Q_i \in \mathcal{C}$ contains exactly one vertex of $N(x)$. By the symmetry of G_x, we may, without loss of generality, assume that $V(Q_i) \cap V(G_x) = \{v_i\}$ for each $i \in [6]$. Thus, in G, there is a K_6-subdivision H with branch vertices v_1, v_2, v_4, v_5, x and z. The paths in H connecting the branch vertices of are as indicated in Figure 3. Note that the (x, z)-path in H is the union of the (z, v_3)-path Q_3 and the (v_3, x)-path $(\{v_3, x\}, \{v_3 x\})$. Thus, G contains a subdivision of K_6. □

The following conjecture, known as the $(k-1, 1)$ *Minor Conjecture*, is a well-known relaxed version of Hadwiger's Conjecture.

Conjecture 7.5 (Chartrand, Geller & Hedetniemi [5]; Woodall [33]). *Every* k*-chromatic graph has either a* K_k *minor or a* $K_{\lfloor \frac{k+1}{2} \rfloor, \lceil \frac{k+1}{2} \rceil}$ *minor.*

Kawarabayashi and Toft [16] proved that every 7-chromatic graph contains K_7 or $K_{4,4}$ as a minor – thus, settling the case $k = 7$ of the $(k-1, 1)$ Minor Conjecture. This result has inspired the following problem.

Problem 7.6. *Prove that every double-critical* 8*-chromatic graph contains* K_8 *or* $K_{4,5}$ *as a minor.*

[3] Private communication to the author, Odense, September, 2008.

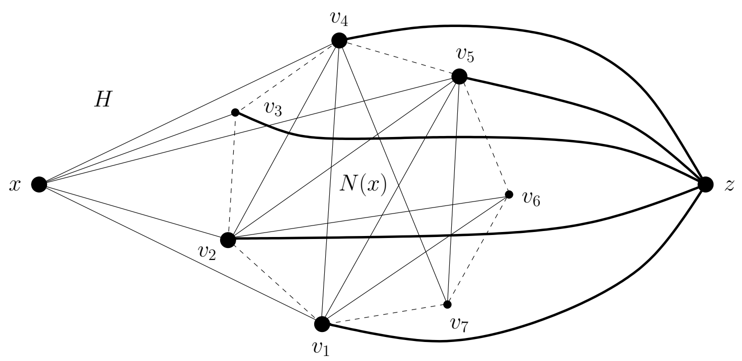


Figure 3: The graph H of G is a subdivision of K_6. The six larger dots represent the branch vertices of H, while the smaller dots represent subdividing vertices. The filled straight lines represent edges in H, while the bold curves represent the paths Q_1, Q_2, Q_3, Q_4, and Q_5.

A natural generalisation of Problem 7.1 would be to ask for a linear function f such that every double-critical k-chromatic graph has a clique of order $f(k)$; if that problem is too hard it might be worth considering the following problem.

Problem 7.7 (Sergey Norin[4]). *Prove that there a linear, strictly increasing function f such that every double-critical k-chromatic graph has a complete minor of order $f(k)$.*

Acknowledgement

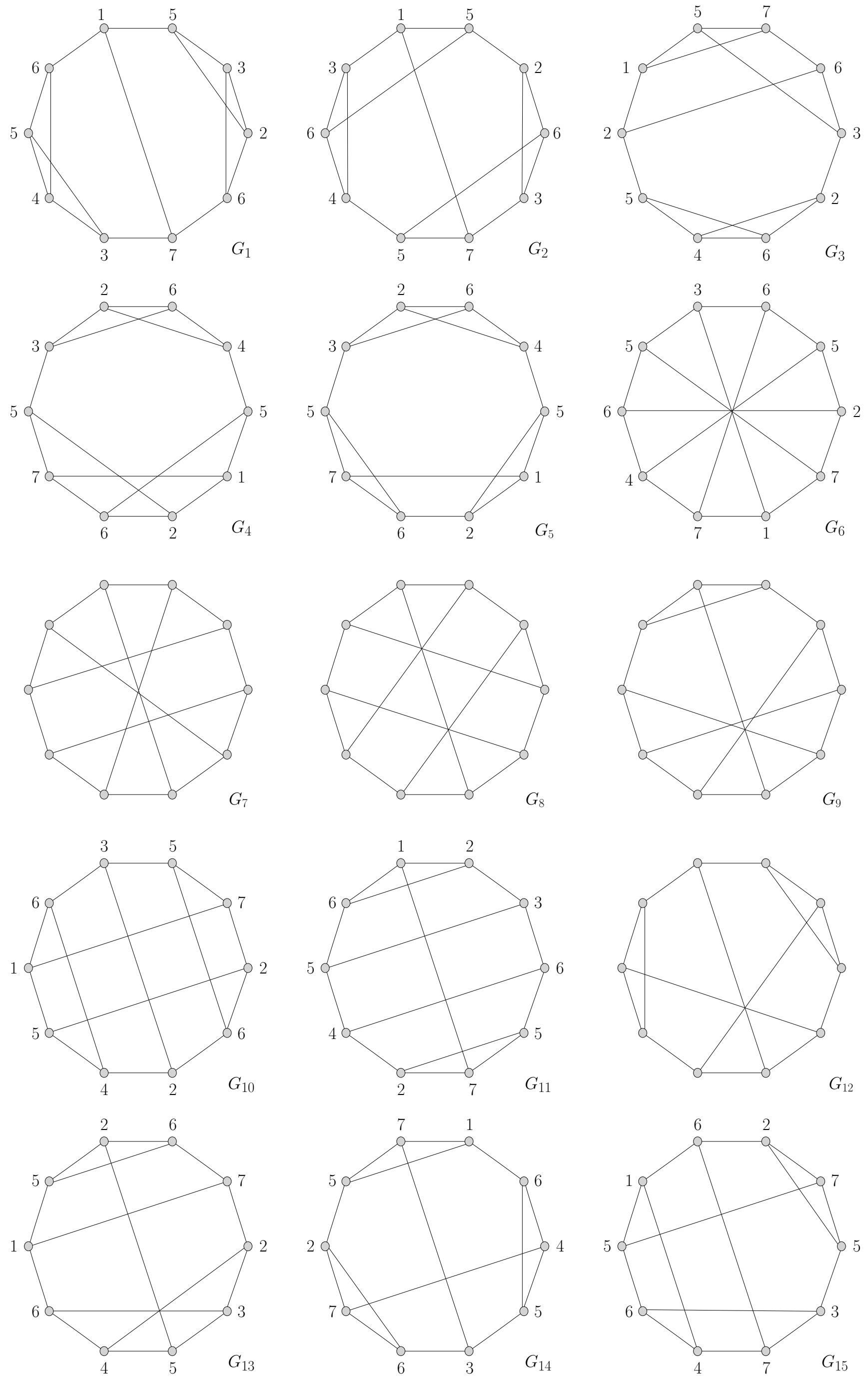
I wish to thank Marco Chiarandini, Daniel Merkle, Friedrich Regen, and Bjarne Toft for stimulating discussions on critical graphs and for assistance in using certain computer programs, in particular, I must thank Friedrich and Marco for developing certain computer programs for sorting and displaying small graphs.

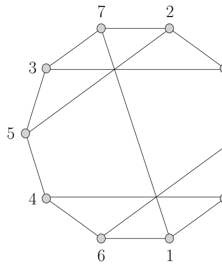
Appendix A

This section contains drawings of all non-isomorphic cubic graphs G_i ($i \in [21]$) of order 10 - the drawings are copies of drawings found in [18]. Drawings of all non-isomorphic cubic graphs of order at most 14 be found in [23].

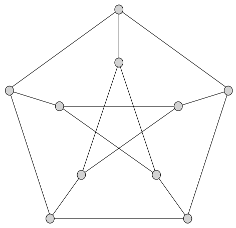
For $i \in [19] \setminus \{7, 8, 9, 12, 17\}$, the labelling of the vertices of the graph G_i indicates how $\overline{G_i}$ may be contracted to K_7^- or, even, K_7. The vertices labelled $j \in [7]$ constitute the jth branch set of a K_7^-- or K_7 minor. If the branch sets only constitute a K_7^- minor, then it is because there is no edge between the branch sets of vertices labelled 1 and 7, respectively.

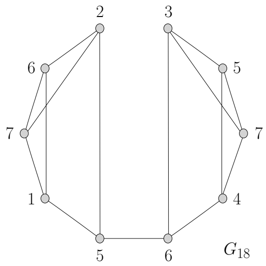
[4]Private communication to the author at Prague Midsummer Combinatorial Workshop XV, July 27 - July 31, 2009.


G_1
G_2
G_3
G_4
G_5
G_6
G_7
G_8
G_9
G_{10}
G_{11}
G_{12}
G_{13}
G_{14}
G_{15}

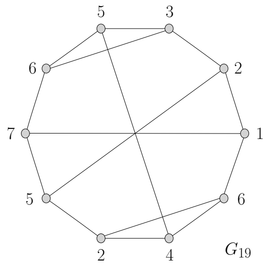


G_{16}

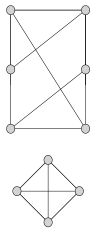

G_{17}

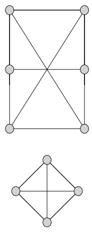


G_{18}



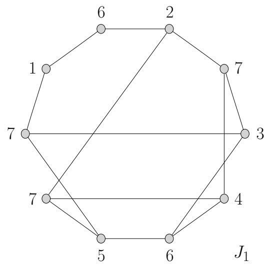
G_{19}


G_{20}

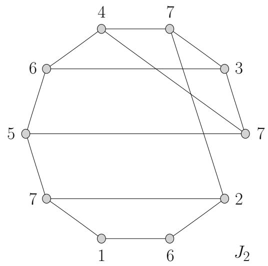

G_{21}

Appendix B

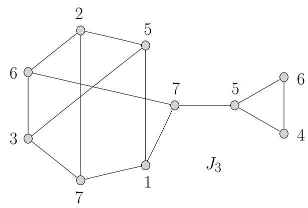
This appendix depicts 23 graphs J_i ($i \in [23]$). The vertices of each graph J_i ($i \in [23]$) are labelled with the integers 1 to 7 such that the vertices labelled $j \in [7]$ constitute the jth branch set of a K_7^-- or K_7 minor. If the branch sets only constitute a K_7^- minor, then it is because there is no edge between the branch sets of vertices labelled 1 and 7, respectively.



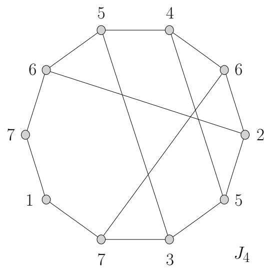
J_1



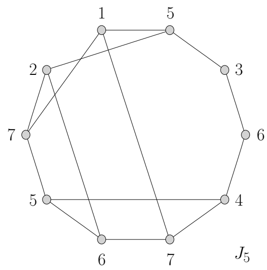
J_2



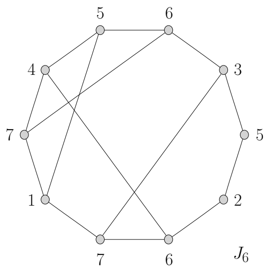
J_3



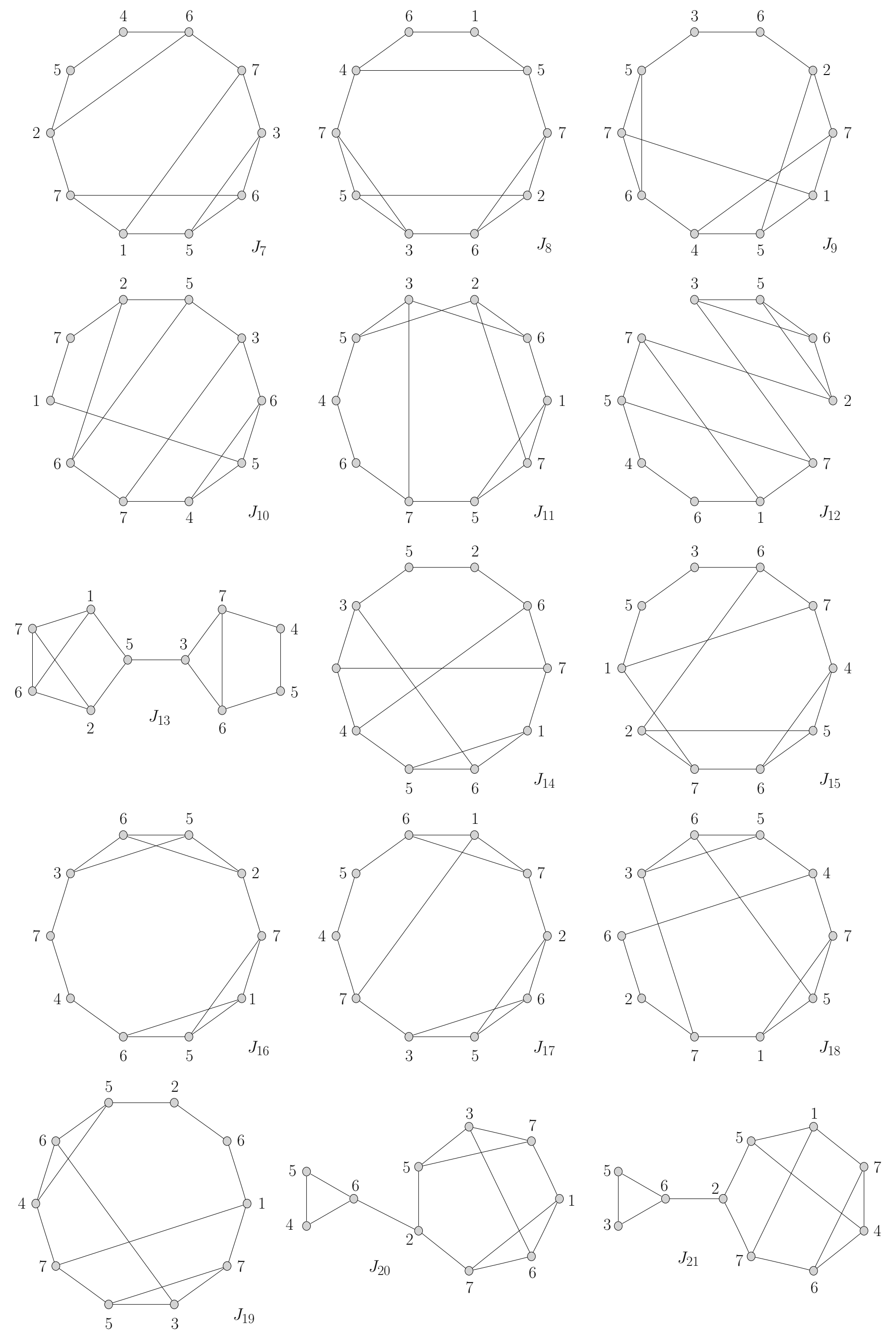
J_4

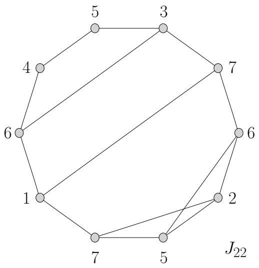


J_5

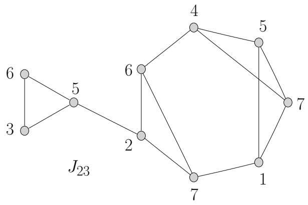


J_6


J_7
J_8
J_9
J_{10}
J_{11}
J_{12}
J_{13}
J_{14}
J_{15}
J_{16}
J_{17}
J_{18}
J_{19}
J_{20}
J_{21}



J_{22}



J_{23}

References

[1] K. Appel and W. Haken. Every planar map is four colorable. I. Discharging. *Illinois J. Math.*, 21(3):429–490, 1977.

[2] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable. II. Reducibility. *Illinois J. Math.*, 21(3):491–567, 1977.

[3] B. Bollobás. *Extremal graph theory*, volume 11 of *London Mathematical Society Monographs.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1978.

[4] J. A. Bondy and U. S. R. Murty. *Graph theory*, volume 244 of *Graduate Texts in Mathematics.* Springer, New York, 2008.

[5] G. Chartrand, D. Geller, and S. Hedetniemi. Graphs with forbidden subgraphs. *J. Combinatorial Theory Ser. B*, 10:12–41, 1971.

[6] R. Diestel. *Graph theory. 3rd revised and extended ed.* Berlin: Springer, 2006.

[7] G. A. Dirac. A contraction theorem for abstract graphs. *Math. Ann.*, 144: 93–96, 1961.

[8] G. A. Dirac. On the structure of 5- and 6-chromatic abstract graphs. *J. Reine Angew. Math.*, 214/215:43–52, 1964.

[9] P. Erdős. Problem 2. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, page 361. Academic Press, New York, 1968.

[10] H. Hadwiger. Über eine Klassifikation der Streckenkomplexe. *Vierteljschr. Naturforsch. Ges. Zürich*, 88:133–142, 1943.

[11] I. T. Jakobsen. *On certain homomorphism-properties of graphs with applications to the conjecture of Hadwiger.* Matematisk Institut, Aarhus Universitet, Aarhus, 1970. Doctoral dissertation, University of London, 1971, Various Publications Series, No. 15.

[12] I. T. Jakobsen. A homomorphism theorem with an application to the conjecture of Hadwiger. *Studia Sci. Math. Hungar.*, 6:151–160, 1971.

[13] I. T. Jakobsen. On certain homomorphism properties of graphs. I. *Math. Scand.*, 31:379–404, 1972.

[14] T. R. Jensen and B. Toft. *Graph coloring problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1995. A Wiley-Interscience Publication.

[15] L. K. Jørgensen. Contractions to K_8. *J. Graph Theory*, 18(5):431–448, 1994.

[16] K. Kawarabayashi and B. Toft. Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor. *Combinatorica*, 25(3):327–353, 2005.

[17] K. Kawarabayashi, A. S. Pedersen, and B. Toft. Double-critical graphs and complete minors. *Electron. J. Combin.*, 17(1):Research Paper 87, 27 pp., 2010.

[18] G. B. Khosrovshahi, Ch. Maysoori, and B. Tayfeh-Rezaie. A note on 3-factorizations of K_{10}. *J. Combin. Des.*, 9(5):379–383, 2001.

[19] W. Mader. Homomorphiesätze für Graphen. *Math. Ann.*, 178:154–168, 1968.

[20] W. Mader. $3n - 5$ edges do force a subdivision of K_5. *Combinatorica*, 18 (4):569–595, 1998.

[21] B. McKay. The nauty page. `http://cs.anu.edu.au/~bdm/nauty/`, 2009.

[22] N. N. Mozhan. Twice critical graphs with chromatic number five. *Metody Diskret. Analiz.*, (46):50–59, 73, 1987.

[23] R. C. Read and R. J. Wilson. *An atlas of graphs*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1998.

[24] N. Robertson, P. Seymour, and R. Thomas. Hadwiger's conjecture for K_6-free graphs. *Combinatorica*, 13(3):279–361, 1993.

[25] N. Robertson, D. Sanders, P. Seymour, and R. Thomas. The four-colour theorem. *J. Combin. Theory Ser. B*, 70(1):2–44, 1997.

[26] Z. Song. The extremal function for K_8^- minors. *J. Combin. Theory Ser. B*, 95(2):300–317, 2005.

[27] M. Stiebitz. K_5 is the only double-critical 5-chromatic graph. *Discrete Math.*, 64(1):91–93, 1987.

[28] M. Stiebitz. On k-critical n-chromatic graphs. In *Combinatorics (Eger, 1987)*, volume 52 of *Colloq. Math. Soc. János Bolyai*, pages 509–514. North-Holland, Amsterdam, 1988.

[29] B. Toft. Colouring, stable sets and perfect graphs. In *Handbook of combinatorics, Vol. 1, 2*, pages 233–288. Elsevier, Amsterdam, 1995.

[30] B. Toft. A survey of Hadwiger's conjecture. *Congr. Numer.*, 115:249–283, 1996.

[31] K. Wagner. Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.*, 114 (1):570–590, 1937.

[32] K. Wagner. Bemerkungen zu Hadwigers Vermutung. *Math. Ann.*, 141: 433–451, 1960.

[33] D. Woodall. Improper colourings of graphs. In *Graph colourings (Milton Keynes, 1988)*, volume 218 of *Pitman Res. Notes Math. Ser.*, pages 45–63. Longman Sci. Tech., Harlow, 1990.